

Testing additivity in nonparametric regression under random censorship

Mohammed DEBBARH ^a

^a*L.S.T.A.-Université Paris 6, 175, rue du Chevaleret, 75013, Paris.
debbahr@ccr.jussieu.fr*

Vivian VIALLO ^{a,b}

^b*Department of Biostatistics, Hôpital Cochin, Université Paris-Descartes, 27 rue du Faubourg Saint Jacques 75014 Paris. vivian.viallon@univ-paris5.fr*

Abstract

In this paper, we are concerned with nonparametric estimation of the multivariate regression function in the presence of right censored data. More precisely, we propose a statistic that is shown to be asymptotically normally distributed under the additive assumption, and that could be used to test for additivity in the censored regression setting.

Key words: censored regression; additive model; curse of dimensionality; nonparametric regression; marginal integration.

1 Introduction and motivations

A well known issue in nonparametric regression estimation is the so-called *curse of dimensionality*, i.e. the fact that the rate of convergence of nonparametric estimators dramatically decreases as the dimension of the covariates increases (see, for instance, Stone (1982)). To get round this issue, one common solution is to work under the *additive assumption*, i.e. the true regression function is assumed to be the sum of some lower dimension regression functions (typically, univariate or bivariate functions). But, this assumption is strong and has therefore to be checked *via* one of the available tests (Camlong-Viot (2001), Gozalo and Linton (2001), Sperlich *et al.* (2002), Derbort *et al.* (2002)) before being used in practice.

When the variable of interest is censored, several nonparametric estimators have been proposed for the multivariate regression function (see, e.g., Fan and Gijbels (1994), Carbonez *et al.* (1995), Kohler *et al.* (2002), Brunel and Comte (2006)).

By combining one of this 'initial' estimator with the marginal integration method (see Newey (1994), Linton and Nielsen (1995)), estimates can be obtained under the additive assumption. In particular, Debbbarh and Viallon (2007) made use of an initial *Inverse Probability of Censoring Weighted* estimator (such as the one proposed by Carbonez *et al.* (1995)), and established the uniform convergence rate for the corresponding additive estimator. However, in this censored setting, no test for additivity has been proposed yet. That will be our concern here. Namely, we first exhibit a statistic evaluating a weighted difference between the observations of the variable of interest and the estimator we derive *via* the marginal integration method. Then, this statistic is shown to be asymptotically normally distributed under the additive assumption.

To build our estimators, and then our test statistic, the following notations are needed. Let (Y, C, \mathbf{X}) , (Y_1, C_1, \mathbf{X}_1) , $(Y_2, C_2, \mathbf{X}_2), \dots$ be independent and identically distributed $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ -valued random variables. Here Y is the variable of interest, C the censoring variable and $\mathbf{X} = (X_1, \dots, X_d)$ a vector of concomitant variables. In the right censorship model, the only available information on (Y, C) is given by (Z, δ) , with $Z = \min\{Y, C\}$ and $\delta = \mathbb{I}_{\{Y \leq C\}}$, \mathbb{I}_E standing for the indicator function of the set E . As a matter of fact, the observed sample is $\mathcal{D}_n := (\mathbf{X}_i, Z_i, \delta_i)_{1 \leq i \leq n}$, for a given $n \geq 1$.

Given a real measurable function ψ , our concern here is the regression function of $\psi(Y)$ evaluated at $\mathbf{X} = \mathbf{x}$, that is,

$$m_\psi(\mathbf{x}) = E(\psi(Y) \mid \mathbf{X} = \mathbf{x}), \quad \forall \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d. \quad (1.1)$$

Under the traditional additive assumption, the regression function defined in (1.1) can be written as the sum of some (unknown) univariate regression functions m_l ,

$$m_\psi(\mathbf{x}) = m_{\psi, add}(\mathbf{x}) := \mu + \sum_{l=1}^d m_l(x_l). \quad (1.2)$$

In view of (1.2), the functions m_l , as well as the constant term μ , are defined up to an additive constant. Therefore, we will work under the common identifiability condition $E m_l(X_l) = 0$, for $l = 1, \dots, d$. This condition implies that $\mu = E(\psi(Y))$.

In the sequel, we set, for all $t \in \mathbb{R}$, $F(t) = P(Y > t)$, $G(t) = P(C > t)$ and $H(t) = P(Z > t)$ the survival functions pertaining to Y , C and Z respectively. Further denote by G_n the Kaplan-Meier (Kaplan and Meier (1958)) estimator of G ,

$$G_n(y) = \prod_{1 \leq i \leq n} \left(\frac{N_n(Z_i) - 1}{N_n(Z_i)} \right)^{\beta_i}, \quad \text{for all } y \geq 0, \quad \text{with } \beta_i = \mathbb{I}_{\{Z_i \leq y\}}(1 - \delta_i). \quad (1.3)$$

Here we defined $N_n(x) = \sum_{i=1}^n \mathbb{I}_{\{Z_i \leq x\}}$, and the conventions $\prod_{\emptyset} = 1$ and $0^0 = 1$ were adopted.

Consider the null hypothesis

$$H_0 : m_\psi \in \mathcal{M}_{add} := \{m : \mathbb{R}^d \rightarrow \mathbb{R}, m(\mathbf{x}) = \mu + \sum_{l=1}^d m_l(x_l); E(m_l(X_l)) = 0\}.$$

Following the ideas of Härdle and Mammen (1993), Camlong-Viot (2001) and González-Manteiga *et al.* (2002), we denote by g some fixed weight function, by L a given *kernel*, i.e. a real measurable function integrating to 1, defined in \mathbb{R}^d and by $(\ell_n)_{n \geq 1}$ a sequence of positive real numbers. Further let $\widehat{m}_{\psi,add}^*$ be some estimator of m_ψ under the additive assumption (1.2) (or, equivalently, under H_0). Now, let us consider the statistic

$$T_n^* = \int_{\mathbb{R}^d} \left[\frac{1}{n\ell_n^d} \sum_{i=1}^n L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) \left(\frac{\frac{\delta_i \psi(Z_i)}{G_n(Z_i)} - \widehat{m}_{\psi,add}^*(\mathbf{X}_i)}{\widehat{f}_n(\mathbf{X}_i)} \right)^2 \right] g(\mathbf{x}) d\mathbf{x}, \quad (1.4)$$

which is a natural estimator of the quantity $E(E(\frac{\delta \psi(Z)}{G(Z)} - m_{\psi,add}(\mathbf{X})) | \mathbf{X})^2$. Under a useful independence condition (see (C.1) below), the latter quantity equals $E(E(\psi(Y) - m_{\psi,add}(\mathbf{X})) | \mathbf{X})^2)$, and then equals zero if and only if the hypothesis H_0 is true. Moreover, in Theorem 2.1 below, this statistic is shown to be asymptotically normally distributed under H_0 . Therefore, it could be useful to test for additivity in censored nonparametric regression. Properties of the corresponding test will be studied elsewhere.

Now, we precise how $\widehat{m}_{\psi,add}^*$ may be constructed. Let K_1, K_2, K_3 and K , be kernels respectively defined in $\mathbb{R}, \mathbb{R}^{d-1}, \mathbb{R}^d$ and \mathbb{R}^d . Further set \widehat{f}_n the kernel estimator of f , with f denoting the density function of \mathbf{X} . Namely,

$$\widehat{f}_n(\mathbf{x}) = \frac{1}{nh_n^d} \sum_{j=1}^n K\left(\frac{\mathbf{X}_j - \mathbf{x}}{h_n}\right),$$

where $(h_n)_{n \geq 1}$ is a given sequence of positive real numbers. Denote by $(h_{j,n})_{n \geq 1}$, $j = 1, 2$, two sequences of positive real numbers. To estimate the multivariate regression function defined in (1.1), the following Nadaraya-Watson type estimators can be used (see Carbonez *et al.* (1995), Kohler *et al.* (2002) and Jones *et al.* (1994)),

$$\widetilde{m}_{\psi,n}^*(\mathbf{x}) = \sum_{i=1}^n W_{n,i}(\mathbf{x}) \frac{\delta_i \psi(Z_i)}{G_n(Z_i)} \quad \text{with} \quad W_{n,i}(\mathbf{x}) = \frac{K_3\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_{1,n}}\right)}{nh_{1,n}^d \widehat{f}_n(\mathbf{X}_i)}, \quad (1.5)$$

and, for $l = 1, \dots, d$,

$$\widetilde{m}_{\psi,n,l}^*(\mathbf{x}) = \sum_{i=1}^n W_{n,i}^l(\mathbf{x}) \frac{\delta_i \psi(Z_i)}{G_n(Z_i)} \quad \text{with} \quad W_{n,i}^l(\mathbf{x}) = \frac{K_1\left(\frac{x_l - X_{i,l}}{h_{1,n}}\right) K_2\left(\frac{\mathbf{x}_{-l} - \mathbf{X}_{i,-l}}{h_{2,n}}\right)}{nh_{1,n} h_{2,n}^{d-1} \widehat{f}_n(\mathbf{X}_i)}, \quad (1.6)$$

where we set, for all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and every $l = 1, \dots, d$, $\mathbf{x}_{-l} = (x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_d)$. To estimate the additive components, we use the marginal integration method (see Newey (1994) or Linton and Nielsen (1995)). Let q_1, \dots, q_d be d given density functions. Then, setting $q(\mathbf{x}) = \prod_{l=1}^d q_l(x_l)$ and $q_{-l}(\mathbf{x}_{-l}) = \prod_{j \neq l} q_j(x_j)$, we define

$$\eta_l(x_l) = \int_{\mathbb{R}^{d-1}} m_\psi(\mathbf{x}) q_{-l}(\mathbf{x}_{-l}) d\mathbf{x}_{-l} - \int_{\mathbb{R}^d} m_\psi(\mathbf{x}) q(\mathbf{x}) d\mathbf{x}, \quad l = 1, \dots, d, \quad (1.7)$$

in such a way that the two following equalities hold,

$$\eta_l(x_l) = m_l(x_l) - \int_{\mathbb{R}} m_l(z) q_l(z) dz, \quad l = 1, \dots, d, \quad (1.8)$$

$$m_\psi(\mathbf{x}) = \sum_{l=1}^d \eta_l(x_l) + \int_{\mathbb{R}^d} m_\psi(\mathbf{z}) q(\mathbf{z}) d\mathbf{z}. \quad (1.9)$$

In view of (1.8) and (1.9), the functions η_l , $l = 1, \dots, d$, turn out to be some additive components, and, from (1.6) and (1.7), a natural estimator of the l -th component η_l is given, for all $l = 1, \dots, d$, by

$$\hat{\eta}_l^*(x_l) = \int_{\mathbb{R}^{d-1}} \hat{m}_{\psi,n,l}^*(\mathbf{x}) q_{-l}(\mathbf{x}_{-l}) d\mathbf{x}_{-l} - \int_{\mathbb{R}^d} \hat{m}_{\psi,n,l}^*(\mathbf{x}) q(\mathbf{x}) d\mathbf{x}. \quad (1.10)$$

From (1.10), an estimator $\hat{m}_{\psi,add}^*$ of the censored regression function can be deduced under the additive assumption (1.2) (or, equivalently, H_0),

$$\hat{m}_{\psi,add}^*(\mathbf{x}) = \sum_{l=1}^d \hat{\eta}_l^*(x_l) + \int_{\mathbb{R}^d} \hat{m}_{\psi,n}^*(\mathbf{x}) q(\mathbf{x}) d\mathbf{x}. \quad (1.11)$$

2 Hypotheses and Results

These preliminaries being given, we introduce the assumptions to be made to state our results. First, consider the hypotheses pertaining to (Y, C, \mathbf{X}) . We suppose that (\mathbf{X}, Y) has a joint density $f_{\mathbf{X},Y}$. Moreover, we impose the following conditions.

(C.1) : C and (\mathbf{X}, Y) are independent.

(C.2) : G is continuous.

(C.3) : There exists a constant $M < \infty$ such that $\sup_{0 \leq t \leq \tau} |\psi(t)| \leq M$.

(C.4) : m_ψ is a k -times continuously differentiable function, $k \geq 1$, and

$$\sup_{\mathbf{x}} \left| \frac{\partial^k m_\psi}{\partial x_l^k}(\mathbf{x}) \right| < \infty; \quad l = 1, \dots, d.$$

Remark 1 *It is noteworthy that condition (C.1) is stronger than the conditional independence of C and Y given \mathbf{X} , under which Beran (1981) worked to build an estimator of the conditional survival function (see also Dabrowska*

(1995)). Note, however, that the two assumptions coincide if C and \mathbf{X} are independent. In other respect, to use Beran's local Kaplan-Meier estimator, the censoring has to be locally fair, that is $P[C \geq t \mid \mathbf{X} = \mathbf{x}] > 0$ whenever $P[Y \geq t \mid \mathbf{X} = \mathbf{x}] > 0$. Here (see assumption (A)(ii)(b) below), we essentially suppose that $G(t) > 0$ whenever $F(t) > 0$, which is, on its turn, a weaker assumption. For a nice discussion on the difference between Beran's estimator and Inverse Probability of Censoring Weighted type estimators, we refer to Carbonez et al. (1995).

Denote by $\mathcal{C}_1, \dots, \mathcal{C}_d$, d compact intervals of \mathbb{R} and set $\mathcal{C} = \mathcal{C}_1 \times \dots \times \mathcal{C}_d$. For every subset \mathcal{E} of \mathbb{R}^q , $q \geq 1$, and any $\alpha > 0$, introduce the α -neighborhood \mathcal{E}^α of \mathcal{E} , i.e. $\mathcal{E}^\alpha = \{x : \inf_{y \in \mathcal{E}} \|x - y\|_{\mathbb{R}^q} \leq \alpha\}$, $\|\cdot\|_{\mathbb{R}^q}$ standing for the euclidian norm on \mathbb{R}^q .

We will work under the following regularity assumptions on f and f_l , $l = 1, \dots, d$, f_l denoting the density function of X_l . These functions are supposed to be continuous and we assume the existence of a constant $\alpha > 0$ such that the following assumptions hold,

$$(F.1) : \forall x_l \in \mathcal{C}_l^\alpha, f_l(x_l) > 0, \quad l = 1, \dots, d, \quad \text{and} \quad \forall \mathbf{x} \in \mathcal{C}^\alpha f(\mathbf{x}) > 0.$$

$$(F.2) : f \text{ is } k'\text{-times continuously differentiable on } \mathcal{C}^\alpha, k' > kd.$$

Regarding the weight function g , we will assume that the condition (G.1) below is satisfied.

$$(G.1) : g \text{ is an indicator function with compact support included in } \mathcal{C}.$$

Remark 2.1 *Assumption (G.1) is made here to avoid technical issues in the derivation of our results. Moreover, it is not restrictive since g is a given weight function. That being said, this assumption could be relaxed (see, e.g., González-Manteiga et al. (2002), Camlong-Viot (2001)).*

The kernels L , K , and K_3 defined in \mathbb{R}^d , K_1 defined in \mathbb{R} and K_2 defined in \mathbb{R}^{d-1} , are assumed to be continuous, compactly supported and integrating to 1. Moreover, we suppose that,

$$(K.1) : K_1 \text{ is Lipschitz;}$$

$$(K.2) : K_1 \text{ and } K_3 \text{ are of order } k, \text{ and } K \text{ is of order } k'.$$

In addition, we impose the following assumptions on the integrating density functions q_{-l} and q_l , $l = 1, \dots, d$.

$$(Q.1) : q_{-l} \text{ is bounded and continuous, } l = 1, \dots, d.$$

$$(Q.2) : q_l \text{ has } k+1 \text{ continuous and bounded derivatives, } l = 1, \dots, d.$$

Turning our attention to the smoothing parameters ℓ_n , h_n and $h_{j,n}$, $j = 1, 2$, we will work under the conditions below.

$$(H.1) : h_n = c_1 \left(\frac{\log n}{n} \right)^{1/(2k'+d)}, \text{ for a given } 0 < c_1 < \infty.$$

$$(H.2) : h_{1,n} = c_2 \left(\frac{\log n}{n} \right)^{1/(2k+1)}, \text{ for a given } 0 < c_2 < \infty \text{ and } h_{2,n} = o(1).$$

$$(H.3) : n(\log n/n)^{k/(2k+1)} \ell_n^{d/2} \rightarrow 0 \text{ and } n \ell_n^d \rightarrow \infty.$$

As mentioned in Gross and Lai (1996), functionals of the (conditional) law can generally not be estimated on the complete support when the variable of interest is right-censored. Accordingly, we will work under the assumption **(A)** that will be said to hold if either **(A)**(i) or **(A)**(ii) below holds. Denote by $T_L = \sup\{t : L(t) > 0\}$ the upper endpoint of the distribution of a random variable with right continuous survival function L .

(A)(i) There exists a $\tau_0 < T_H$ such that $\psi = 0$ on (τ_0, ∞) .

(A)(ii) (a) For a given $k/(2k+1) < p \leq 1/2$, $\left| \int_0^{T_H} F^{-p/(1-p)} dG \right| < \infty$;
 (b) $T_F < T_G$;
 (c) $n^{2p-1} h_{l,n}^{-1} |\log(h_{l,n})| \rightarrow \infty$, as $n \rightarrow \infty$, for every $l = 1, \dots, d$.

It is noteworthy that assumption **(A)**(ii) allows for considering the estimation of the "classical" regression function, which corresponds to the choice $\psi(y) = y$. On the other hand, normality for estimators of functionals such as the conditional distribution function $P(Y \leq \tau_0 | \mathbf{X})$ can be obtained under weaker conditions, when restricting ourselves to $\tau_0 < T_H$.

To state our result, some additional notations are needed. Set $\epsilon_i = \frac{\delta_i \psi(Z_i)}{G(Z_i)} - m_\psi(\mathbf{X}_i)$ and $\sigma_0^2(\mathbf{x}) = E(\epsilon_i^2 | \mathbf{X}_i = \mathbf{x})$. Further introduce $B = [\int \sigma_0^2(\mathbf{u}) f^{-1}(\mathbf{u}) g(\mathbf{u}) d\mathbf{u}] \times \int L^2(\mathbf{t}) d\mathbf{t}$ and $V = 2 [\int (\sigma_0^2(\mathbf{u})^2 f^{-2}(\mathbf{u}) g^2(\mathbf{u}) d\mathbf{u}) \times \int [\int L(\mathbf{t}) L(\mathbf{t} - \mathbf{r}) d\mathbf{t}]^2 d\mathbf{r}]$.

Theorem 2.1 *Assume the conditions **(A)**, (C.1-4), (F.1-2), (G.1), (K.1-2), (Q.1-2) and (H.1-3) hold. Then, under the null hypothesis H_0 , we have,*

$$\frac{n \ell_n^{d/2} T_n^* - B \ell_n^{-d/2}}{\sqrt{V}} \rightarrow \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty.$$

3 Proof of Theorem 2.1

Here, we present the proof of Theorem 2.1 in the case where **(A)**(i) holds. The case where **(A)**(ii) holds follows from similar arguments (especially replacing the result of Földes and Rejtő (1981) by that of Gu and Lai (1990) or that of Chen and Lo (1997)); details are then omitted.

We will make frequent use of the following lemma, which was established in Debbarh and Viallon (2007).

Lemma 3.1 *Assume H_0 holds. Then, under the conditions (A), (C.1-4), (F.1-2), (K.1-2), (Q.1-2) and (H.1-2), we have, with probability one,*

$$\sup_{\mathbf{x} \in \mathcal{C}} |\widehat{m}_{\psi,add}^*(\mathbf{x}) - m_{\psi}(\mathbf{x})| = \mathcal{O}\left(\left(\frac{\log n}{n}\right)^{\frac{k}{2k+1}}\right). \quad (3.1)$$

We will also make frequent use of the following result, due to Földes and Rejtő (1981).

$$\text{For all } \tau' < T_H, \quad \sup_{y \leq \tau'} |G_n^*(y) - G(y)| = \mathcal{O}((\log \log n/n)^{1/2}) =: \rho_n \quad (3.2)$$

From this last result, we especially get the following type of approximations. Set $\epsilon_i^* = \frac{\delta_i \psi(Z_i)}{G_n(Z_i)} - m_{\psi}(\mathbf{X}_i)$. Then, from (A)(i), (C.2), (C.3) and (3.2), we have, almost surely as $n \rightarrow \infty$,

$$\epsilon_i^* = \epsilon_i + \mathcal{O}(\rho_n). \quad (3.3)$$

Now, recalling the definition (1.4) of T_n^* , we have

$$T_n^* = \int_{\mathbb{R}^d} \left[\frac{1}{n \ell_n^d} \sum_{i=1}^n L\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right) \left(\frac{m_{\psi}(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i) + \epsilon_i^*}{f_n(\mathbf{X}_i)} \right) \right]^2 g(\mathbf{x}) d\mathbf{x}.$$

Consider the quantity

$$\begin{aligned} T_n^{1*} &= \int_{\mathbb{R}^d} \left[\frac{1}{n \ell_n^d} \sum_{i=1}^n L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) \left(\frac{m_{\psi}(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i) + \epsilon_i^*}{f(\mathbf{X}_i)} \right) \right]^2 g(\mathbf{x}) d\mathbf{x}. \\ &= T_{n,1}^{1*} + T_{n,2}^{1*} + T_{n,3}^{1*} + 2T_{n,4}^{1*} + 2T_{n,5}^{1*}, \end{aligned}$$

with (see Camlong-Viot (2001)),

$$\begin{aligned} T_{n,1}^{1*} &= \int_{\mathbb{R}^d} \frac{1}{n^2 \ell_n^{2d}} \sum_{i=1}^n L^2\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) \frac{\epsilon_i^{*2} g(\mathbf{x})}{f^2(\mathbf{X}_i)} d\mathbf{x}, \\ T_{n,2}^{1*} &= \int_{\mathbb{R}^d} \frac{1}{n^2 \ell_n^{2d}} \sum_{i \neq j} L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) L\left(\frac{\mathbf{x} - \mathbf{X}_j}{\ell_n}\right) \frac{\epsilon_i^* \epsilon_j^* g(\mathbf{x})}{f(\mathbf{X}_i) f(\mathbf{X}_j)} d\mathbf{x}, \\ T_{n,3}^{1*} &= \int_{\mathbb{R}^d} \left[\frac{1}{n \ell_n^d} \sum_{i=1}^n L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) \left(\frac{m_{\psi}(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i)}{f(\mathbf{X}_i)} \right) \right]^2 g(\mathbf{x}) d\mathbf{x}, \\ T_{n,4}^{1*} &= \int_{\mathbb{R}^d} \left[\frac{1}{n \ell_n^d} \right]^2 \sum_{i=1}^n L^2\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) (m_{\psi}(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i)) \epsilon_i^* \frac{g(\mathbf{x})}{f^2(\mathbf{X}_i)} d\mathbf{x}, \\ T_{n,5}^{1*} &= \int_{\mathbb{R}^d} \left[\frac{1}{n \ell_n^d} \right]^2 \sum_{i \neq j} L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) L\left(\frac{\mathbf{x} - \mathbf{X}_j}{\ell_n}\right) \left(\frac{m_{\psi}(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i)}{f(\mathbf{X}_i) f(\mathbf{X}_j)} \right) \epsilon_j^* g(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

By (F.1-2), (H.1) and (K.2), it holds that, almost surely as $n \rightarrow \infty$ (see for instance Ango-Nze and Rios (2000)),

$$\sup_{\mathbf{x} \in \mathcal{C}} |\hat{f}_n(\mathbf{x}) - f(\mathbf{x})| = \mathcal{O}\left(\sqrt{\frac{\log n}{nh_n^d}}\right), \quad (3.4)$$

in such a way that $T_n^* = T_n^{1*} \{1 + \mathcal{O}[(n^{-1}h_n^{-d} \log n)^{1/2}]\}$ almost surely as $n \rightarrow \infty$. Therefore, to achieve the proof of Theorem 2.1, it is sufficient to establish (3.5)–(3.9) below.

$$n\ell_n^d T_{n,1}^{1*} = B + o_p(\ell_n^{d/2}), \quad (3.5)$$

$$n\ell_n^{d/2} \frac{(T_{n,2}^{1*} - ET_{n,2}^{1*})}{\sqrt{V}} \rightarrow N(0, 1), \quad (3.6)$$

$$T_{n,3}^{1*} = o_p(n^{-1}\ell_n^{-d/2}), \quad (3.7)$$

$$T_{n,4}^{1*} = o_p(n^{-1}\ell_n^{-d/2}), \quad (3.8)$$

$$T_{n,5}^{1*} = o_p(n^{-1}\ell_n^{-d/2}). \quad (3.9)$$

Proof of (3.5): Set $\sigma_0^{*2}(\mathbf{x}) = E(\epsilon_i^{*2} | \mathbf{X}_i = \mathbf{x})$. Using a conditioning argument, it is straightforward that

$$ET_{n,1}^{1*} = \frac{1}{n\ell_n^{2d}} \int \int \frac{\sigma_0^{*2}(\mathbf{v})}{f(\mathbf{v})} L^2\left(\frac{\mathbf{x} - \mathbf{v}}{\ell_n}\right) d\mathbf{v} g(\mathbf{x}) d\mathbf{x}.$$

Moreover, since g is an indicator function with compact support included in \mathcal{C} , we obtain, for n large enough, that

$$ET_{n,1}^{1*} = \frac{1}{n\ell_n^d} \int E(\epsilon_1^{*2} | \mathbf{X}_1 = \mathbf{x}) g(\mathbf{x}) f^{-1}(\mathbf{x}) d\mathbf{x} \int L^2(\mathbf{u}) d\mathbf{u} + \mathcal{O}\left(\frac{1}{n}\right),$$

and then, *via* arguments similar to those used to derive (3.3),

$$ET_{n,1}^{1*} = \frac{1}{n\ell_n^d} B + \mathcal{O}(\rho_n), \quad \text{as } n \rightarrow \infty.$$

Turning our attention to the variance of $T_{n,1}^{1*}$, we can write,

$$\text{Var } T_{n,1}^{1*} = \frac{1}{n^3 \ell_n^{4d}} \text{Var}(I), \quad \text{where } I = \int \epsilon_1^{*2} L^2\left(\frac{\mathbf{x} - \mathbf{X}_j}{\ell_n}\right) \frac{g(\mathbf{x})}{f^2(\mathbf{X})} d\mathbf{x}. \quad (3.10)$$

But, using once again the arguments used to show (3.3), along with the facts that $G(\tau) > 0$, ψ and g are bounded and L is compactly supported, it holds that, as $n \rightarrow \infty$,

$$\begin{aligned} E(I^2) &= \int E(\epsilon_1^{*4} | \mathbf{X}_1 = \mathbf{v}) \left(\int L^2\left(\frac{\mathbf{x} - \mathbf{v}}{\ell_n}\right) \frac{g(\mathbf{x})}{f^2(\mathbf{v})} d\mathbf{x} \right)^2 f(\mathbf{v}) d\mathbf{v}, \\ &= \mathcal{O}(\ell_n^{2d}) (1 + \mathcal{O}(\rho_n)). \end{aligned} \quad (3.11)$$

By (3.10) and (3.11), it follows that $\text{Var } T_{n,1}^{1*} = \mathcal{O}(n^{-3}\ell_n^{-2d})$. Using the Bienayme-Tchebychev inequality, we infer that, for all $\varepsilon > 0$, $P(n\ell_n^{d/2}|T_{n,1}^{1*} - ET_{n,1}^{1*}| \geq \varepsilon) \leq \varepsilon^{-2}n^2\ell_n^d \times \text{Var } T_{n,1}^{1*} = \mathcal{O}((n\ell_n^d)^{-1})$. Thus, $T_{n,1}^{1*} = (n\ell_n^d)^{-1}B + o_p[(n\ell_n^d)^{-1}]$, which is (3.5).

Proof of (3.6): For $1 \leq i \leq n$, set $\zeta_i = (\mathbf{X}_i, \epsilon_i)$ and $u_i(\mathbf{x}) = L[(\mathbf{x} - \mathbf{X}_i)/\ell_n]$. Further introduce $T_{n,2}^{1*} = \ell_n^{-2d} \int \sum_{i < j} u_i(\mathbf{x})u_j(\mathbf{x})\epsilon_i\epsilon_j g(\mathbf{x})f^{-1}(\mathbf{X}_i)f^{-1}(\mathbf{X}_j)d\mathbf{x}$. Note that, in view of (3.2) and (3.4), the dominated convergence theorem ensures that $T_{n,2}^{1*} = T_{n,2}^{1'} + \mathcal{O}(\rho_n)$ almost surely as $n \rightarrow \infty$, with $T_{n,2}^{1'} := 2n^{-2}T_{n,2}^{1'}$. To establish (3.6), we will make use of a central limit theorem for U-statistics due to Hall (1984). Set $M_n(\zeta_i, \zeta_j) = \ell_n^{-2d} \int u_i(\mathbf{x})u_j(\mathbf{x})\epsilon_i\epsilon_j g(\mathbf{x})f^{-1}(\mathbf{X}_i)f^{-1}(\mathbf{X}_j)d\mathbf{x}$ and $N_n(\mathbf{u}, \mathbf{v}) = E(M_n(\zeta_1, \mathbf{u})M_n(\zeta_1, \mathbf{v}))$. To apply Hall's theorem to $T_{n,2}^{1'}$, the conditions [T1], [T2] and [T3] below must be verified.

$$\begin{aligned} [T_1] \quad & E\{M_n(\zeta_1, \zeta_2)|\zeta_1\} = 0. \\ [T_2] \quad & E\{M_n^2(\zeta_1, \zeta_2)\} < \infty. \\ [T_3] \quad & |E\{N_n^2(\zeta_1, \zeta_2)\} + n^{-1}E\{M_n^4(\zeta_1, \zeta_2)\}| / |E\{M_n^2(\zeta_1, \zeta_2)\}|^2 \rightarrow 0. \end{aligned}$$

[T1] is readily satisfied by making use of conditioning arguments. Moreover, arguing as before, the statement [T2] follows from routine analysis. To establish [T3], it is sufficient to prove the results (3.12), (3.13) and (3.14) below.

$$E(N_n^2(\zeta_1, \zeta_2)) = \mathcal{O}(1) \quad \text{as } n \rightarrow \infty, \quad (3.12)$$

$$E(M_n^2(\zeta_1, \zeta_2)) = \ell_n^{-d} \frac{V}{2} + o(\ell_n^{-d}) \quad \text{as } n \rightarrow \infty, \quad (3.13)$$

$$E(M_n^4(\zeta_1, \zeta_2)) = \mathcal{O}(\ell_n^{-3d}) \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

Proof of (3.12): Denote by $f_{\mathbf{X},\epsilon}$ the joint density of (\mathbf{X}, ϵ) (the existence of which being ensured by the assumption (C.1), since (\mathbf{X}, Y) is supposed to have a joint density). It holds that

$$\begin{aligned} E[N_n^2(\zeta_1, \zeta_2)] &= \frac{1}{\ell_n^{8d}} \int \dots \int \left[\int \dots \int L\left(\frac{\mathbf{x}_1 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_1 - \eta_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_1}{\ell_n}\right) \right. \\ &\quad \left. \times L\left(\frac{\mathbf{x}_2 - \eta_2}{\ell_n}\right) \frac{g(\mathbf{x}_1)}{f^2(\mathbf{v}_1)} \frac{g(\mathbf{x}_2)}{f^2(\mathbf{v}_2)} f_{\mathbf{X},\epsilon}(\mathbf{v}_1, e_1) e_1^2 e_2 e_3 d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{x}_1 d\mathbf{x}_2 \right]^2 d\eta_1 d\eta_2. \end{aligned}$$

Using classical changes of variables, together with the assumption (G.1) and the fact that L is compactly supported, (3.12) is straightforward.

Proof of (3.13). Set $\sigma_n^2 = \iint M_n^2(\omega_1, \omega_2) f_{\mathbf{X},\epsilon}(\omega_1) f_{\mathbf{X},\epsilon}(\omega_2) d\omega_1 d\omega_2$. Then,

$$\begin{aligned} \sigma_n^2 &= \int \int \int \int \int \int \frac{1}{\ell_n^{4d}} L\left(\frac{\mathbf{x}_1 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_1 - \mathbf{v}_2}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_2}{\ell_n}\right) \\ &\quad \times e_1^2 e_2^2 f_{\mathbf{X}_1, \epsilon_1}(\mathbf{v}_1, e_1) f_{\mathbf{X}_2, \epsilon_2}(\mathbf{v}_2, e_2) \frac{g(\mathbf{x}_2)}{f^2(\mathbf{v}_2)} \frac{g(\mathbf{x}_1)}{f^2(\mathbf{v}_1)} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{v}_1 d\mathbf{v}_2 de_1 de_2. \end{aligned}$$

Next, noting that $\int e_1^2 f_{\mathbf{X}, \epsilon}(\mathbf{v}_1, e_1) de_1 = E(\epsilon_i^2 | \mathbf{X}_i = \mathbf{v}_1) f(\mathbf{v}_1)$, it follows that

$$\begin{aligned} \sigma_n^2 &= \int \int \int \int \frac{1}{\ell_n^{4d}} L\left(\frac{\mathbf{x}_1 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_1 - \mathbf{v}_2}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_2}{\ell_n}\right) \\ &\times E(\epsilon_i^2 | \mathbf{X}_i = \mathbf{v}_1) f(\mathbf{v}_1) E(\epsilon_i^2 | \mathbf{X}_j = \mathbf{v}_2) f(\mathbf{v}_2) \frac{g(\mathbf{x}_2)}{f^2(\mathbf{v}_2)} \frac{g(\mathbf{x}_1)}{f^2(\mathbf{v}_1)} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{v}_1 d\mathbf{v}_2. \end{aligned}$$

Using the changes of variables, $\mathbf{y}_1 = (\mathbf{x}_1 - \mathbf{v}_1)/\ell_n$, $\mathbf{y}_2 = (\mathbf{x}_2 - \mathbf{v}_1)/\ell_n$ and $\mathbf{r}_1 = (\mathbf{v}_1 - \mathbf{v}_2)/\ell_n$, along with the continuity of f and the dominated convergence theorem, we get,

$$\begin{aligned} \sigma_n^2 &= \int \dots \int \frac{1}{\ell_n^{4d}} L(\mathbf{y}_1) L(\mathbf{y}_2) L(\mathbf{y}_1 - \mathbf{r}_1) L(\mathbf{y}_2 - \mathbf{r}_1) E(\epsilon_i^2 | \mathbf{X}_i = \mathbf{v}_1) \\ &\times E(\epsilon_j^2 | \mathbf{X}_j = \mathbf{v}_1 + \mathbf{r}_1 \ell_n) \frac{g(\mathbf{v}_1 + \mathbf{y}_1 \ell_n)}{f(\mathbf{v}_1)} \frac{g(\mathbf{v}_1 + \mathbf{y}_2 \ell_n)}{f(\mathbf{v}_1 + \mathbf{r}_1 \ell_n)} d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{r}_1 d\mathbf{v}_1, \\ &= \ell_n^{-d} \int \left[\int L(\mathbf{t}) L(\mathbf{t} - \mathbf{r}) d\mathbf{t} \right]^2 d\mathbf{r} \int (E(\epsilon_i^2 | \mathbf{X}_i = \mathbf{r}))^2 \frac{g^2(\mathbf{r})}{f^2(\mathbf{r})} d\mathbf{r} + o(\ell_n^{-d}), \end{aligned}$$

which, recalling the definition of V , is (3.13).

Proof of (3.14): Arguing as before (see also Camlong-Viot (2001)), we can show that, for a given $C < \infty$,

$$\begin{aligned} |E(M_n^4(\zeta_i, \zeta_j))| &\leq \frac{C}{\ell_n^{8d}} \int \dots \int \left| L\left(\frac{\mathbf{x}_1 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_1}{\ell_n}\right) \right. \\ &\quad L\left(\frac{\mathbf{x}_3 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_4 - \mathbf{v}_1}{\ell_n}\right) L\left(\frac{\mathbf{x}_1 - \mathbf{v}_2}{\ell_n}\right) L\left(\frac{\mathbf{x}_2 - \mathbf{v}_2}{\ell_n}\right) \\ &\quad \left. L\left(\frac{\mathbf{x}_3 - \mathbf{v}_2}{\ell_n}\right) L\left(\frac{\mathbf{x}_4 - \mathbf{v}_2}{\ell_n}\right) \right| \frac{g(\mathbf{x}_1)}{f(\mathbf{v}_1)} \frac{g(\mathbf{x}_2)}{f(\mathbf{v}_2)} \\ &\quad \frac{g(\mathbf{x}_3)}{f^2(\mathbf{v}_1)} \frac{g(\mathbf{x}_4)}{f^2(\mathbf{v}_2)} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 \\ &= \mathcal{O}(\ell_n^{-3d}). \end{aligned}$$

Combining (3.12), (3.13) and (3.14), it is readily shown that $[T3]$ holds. Then, Hall's Theorem can be applied to $T_{n,2}^1$. Namely, since $EM_n(\zeta_i, \zeta_j) = 0$, we have $\sqrt{2}(n\sigma_n)^{-1}T_{n,2}^{1'} \rightarrow \mathcal{N}(0, 1)$. Recalling that $T_{n,2}^1 = n^2 T_{n,2}^{1'}/2$, we deduce, from (3.13), that $(n\ell_n^{d/2})T_{n,2}^1/\sqrt{V} \rightarrow \mathcal{N}(0, 1)$. Slutsky's Theorem is now sufficient to conclude to (3.6), since, as already mentioned, $T_{n,2}^{1*} = T_{n,2}^1 + \mathcal{O}(\rho_n)$ almost surely as $n \rightarrow \infty$.

Proof of (3.7): By (G.1),

$$T_{n,3}^{1*} = \sup_{\mathbf{x} \in C} |m_\psi(\mathbf{x}) - \widehat{m}_{\psi, add}^*(\mathbf{x})|^2 \int \left[\frac{1}{n\ell_n^d} \sum_{i=1}^n L\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) \right]^2 g(\mathbf{x}) d\mathbf{x} \quad \text{a.s. .}$$

Since, under H_0 , $m_\psi = m_{\psi,add} \in \mathcal{M}_{add}$, we can apply the result of Lemma 3.1. This latter, when combined with, successively, the boundedness of f , the dominated convergence theorem, Bochner's theorem and the fact that g is compactly supported, yields $T_{n,3}^{1*} = \mathcal{O}\left((\log n/n)^{2k/2k+1}\right)$ almost surely as $n \rightarrow \infty$. Thus, under the assumption (H.3), $T_{n,3}^{1*} = o_p(n^{-1}\ell_n^{-d/2})$ as $n \rightarrow \infty$.

Proof of (3.8): First consider the mean of $T_{n,4}^{1*}$. By (G.1), it holds that

$$|ET_{n,4}^{1*}| \leq \sup_{\mathbf{x} \in C} |m_\psi(\mathbf{x}) - \widehat{m}_{\psi,add}^*(\mathbf{x})| \frac{1}{n\ell_n^{2d}} \int \int \left| E\left(\epsilon_i^* | \mathbf{X}_i = \mathbf{u}\right) L^2\left(\frac{\mathbf{x} - \mathbf{u}}{\ell_n}\right) \right| \times g(\mathbf{x}) d\mathbf{x} d\mathbf{u}.$$

Next, under the assumptions (H.1-2-3), using, successively, the assumption (G.1), the dominated convergence theorem, the equality (3.3), Bochner's theorem and Lemma 3.1, it can be shown that $|ET_{n,4}^{1*}| = \mathcal{O}((n\ell_n^d)^{-1}(\log n/n)^{k/(2k+1)}) = o(n^{-1}\ell_n^{-d/2})$. Turning our attention to the variance of $T_{n,4}^{1*}$, and arguing as before, we get

$$\begin{aligned} \text{Var } T_{n,4}^{1*} &= \frac{1}{n^3\ell_n^{4d}} \text{Var} \left[\int \epsilon_i^* L^2\left(\frac{\mathbf{x} - \mathbf{X}_i}{\ell_n}\right) (m_\psi(\mathbf{X}_i) - \widehat{m}_{\psi,add}^*(\mathbf{X}_i)) \frac{g(\mathbf{x})}{f^2(\mathbf{X}_i)} d\mathbf{x} \right] \\ &= \mathcal{O}\left(\frac{1}{n^3\ell_n^{2d}} \left(\frac{\log n}{n}\right)^{2k/(2k+1)}\right). \end{aligned}$$

An application of Bienayme-Tchebychev's inequality leads to $T_{n,4}^{1*} - ET_{n,4}^{1*} = o_p(n^{-1}\ell_n^{-d/2})$, which implies (3.8), since $|ET_{n,4}^{1*}| = o(n^{-1}\ell_n^{-d/2})$.

Proof of (3.9): Arguing as before, we infer that, ultimately as $n \rightarrow \infty$,

$$ET_{n,5}^{1*} = \mathcal{O}\left(\left(\frac{\log n}{n}\right)^{k/(2k+1)}\right) \quad \text{and} \quad \text{Var } T_{n,5}^{1*} = \mathcal{O}\left(\frac{1}{n^2} \left(\frac{\log n}{n}\right)^{2k/(2k+1)}\right).$$

Therefore, Bienayme-Tchebychev's inequality leads to (3.9).

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